Matrix Probability Problems

A junior college is considering various program changes, and before implementing them, wants to better understand its graduation rate. They compile some data, yielding the conclusions below:

- Each year, about 60% of their first-year complete the year successfully and re-enroll as second-year students the following year. 30% of the students leave the school by the end of the first year, and another 10% re-enroll as first-year students (that is, they repeat the year).
- Each year, about 80% of second-year students complete the year and graduate with an Associate's Degree. About 10% of second-year students re-enroll as second-year students to complete coursework; another 10% leave the school entirely.
- Graduates stay graduates and do not re-enroll; students who leave the college do not drop-outs from this college do not re-enroll in this college.
- This year, there are 500 first-year students and 350 second-year students.

1. Suppose that each year, 5% of students who have left the school return as first-years. (For simplicity's sake, assume that even a second-year student must return as a first-year student after leaving.)
   a. Write a new transition matrix reflecting this change.
   b. What are the absorbing states of the new model?

2. Return to the initial situation (in which students who have left never return). Now start with a class of 500 first-year students, and assume 0 enrollments in all other groups to start. Make a table showing how many students there will be of each type after one year, after two years, and after three years.

3. Is it correct for the college to say that only 30% of its first-years leave school without getting diplomas? Why or why not?

In #4, use the following information:

A simple model for summer weather is that each day is either rainy, cloudy, or sunny. If a particular day is sunny, there is a 50% chance the next day will be sunny, a 30% chance it will be cloudy, and a 20% chance it will be rainy. If a day is cloudy, there is a 40% chance the next day will be sunny, a 30% chance the next day will be rainy, and a 30% chance the next day will be cloudy. Rainy days are followed by sunny days 40% of the time, by cloudy days 30% of the time, and by rainy days 30% of the time.

4. Suppose that the weather report gives an 80% chance of rain on Monday, with 15% chance of clouds and only a 5% chance of sun.
   a. What is the likelihood of each type of weather on Tuesday?
   b. What is the likelihood of each type of weather on Wednesday?
   c. What is the likelihood of each type of weather on Friday?

In 5-8, use the following information. Physicians use matrix models to represent the progress of disease in patients. For example, patients infected with a serious but not lifethreatening illness can be divided into three groups: very sick, somewhat sick, and recovered. Suppose that a patient's state on a given day can be used to predict her state on the next day as follows:

75% of very sick patients stay very sick; 20% become somewhat sick (this is an improvement), and 5% become recovered.
70% of somewhat sick patients stay somewhat sick; 20% recover, while 10% get very sick.
100% of recovered patients stay recovered (there are no relapses).

5. Represent this information as a transition diagram.
6. Represent this information as a transition matrix *T*.

7. True or False
   a. Each column in the matrix sums to 1.
   b. Each row in the matrix sums to 1.
   c. There are no absorbing states.

8. On June 1, a hospital has 100 patients who are very sick, and 0 patients in the other two categories. Let *T* be the transition matrix from problem 6, and let *S* be the matrix below.

$$S = \begin{pmatrix} 100 \\ 0 \\ 0 \end{pmatrix}$$

   a. What does the product *TS* represent?
   b. What does the product *T*(*TS*) represent?
   c. What does the product T^2S represent?

Applying the Mathematics

9. Using the data from first two problems, compute *TS*, T^2S, etc. until the number of students who are enrolled in either year is less than one. What do the remaining entries tell you?
10. The college is considering two different changes. One change would improve counseling during the first year; the second would improve support services for second-year students. The first plan would decrease the percentage of first-year students who leave school to 20% and increase the percentage of first-year students who repeat the first year to 20%. It would not affect second-year students. The second plan would not affect first-year students, but would increase the number of second-year students who graduate to 90%, with only 5% repeating the year (and 5% leaving).

    a. Make a transition matrix for the first plan.
    b. Start with a class of 500 first-year students, and compute *TS*, T^2S, etc. until no students remain in either first or second year. What percent of students eventually graduate?
    c. Now repeat parts a and b with the second plan.
    d. In one or two sentences, explain which plan you think the college should adopt.

11. A rental car agency has three locations: North Side, Downtown, and South Side. Over time, they notice that customers who rent cars at one location often return them to different locations; the car agency wants to determine how the proportion of cars at each

location changes over time. To do so, they use a matrix model in which the probability that a car rented at one location is returned at another is given by the table below:

Returned To \ Rented From	N	D	S
N	0.4	0.3	0.1
D	0.5	0.5	0.6
S	0.1	0.2	0.3

Suppose that initially the cars are evenly distributed between the three locations.

a. Write the distribution matrix for the initial distribution.
b. What will be the distribution after one day?
c. What will be the distribution after five days?
d. What will be the distribution after ten days?
e. Over the long term, does the distribution of cars approach a steady state (or equilibrium)? How do you know?

Answers:

12. Try the following experiment:

a. Make a 2x2 transition matrix with no absorbing states (all entries are positive real numbers less than one; columns sum to 1). Call it *T*.

b. Let $S = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and compute TS, T^2S, etc., until you notice very little change in the results. Write down the value of $\lim_{n\to\infty} T^n S$.

c. Now let $S = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$ and find $\lim_{n\to\infty} T^n S$.

d. Now let $S = \begin{pmatrix} 0.1 \\ 0.9 \end{pmatrix}$ and find $\lim_{n\to\infty} T^n S$.

e. Compare your answers to b, c, and d. What do they tell you about the relationship between the particular values in *S* and the value of the limit?