


Focus Properties of Parabolas and Ellipses—A GSP investigation


Answer all questions on your sketch, with the text tool, and print out after each part. One printout per pair.


Part I—Ellipses


We need to make an ellipse whose foci we can adjust. As with the parabola, we will construct the ellipse using the coordinate plane. So...open GSP and bring up a square grid.

1. Plot a point on the x -axis and label it A . Plot a point between A and the origin and label it F_1 (note: typing **F[1]** in the label box gives a subscript!). Measure the x -coordinates of both points and relabel them a and c .
2. Make a new calculation of $\sqrt{a^2 - c^2}$ and rename it b . Plot the point $(0,b)$ (to do this you have to first make another calculation whose value is just 0, then select the 0 and the b and go to Graph→Plot(x,y).
3. If $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then $y =$ _____ ? Fill in the blank, then plot that as a function (Graph→Plot New Function). Go back and make sure you graph the other half of the ellipse as well. (The easiest way to do this is to define a new function $g(x) = -f(x)$.)
4. Reflect focus F_1 in the y -axis (double-click the y -axis to make it the axis of reflection, then select F_1 and go to Transform→Reflect) and name the reflected point F_2
5. Stop and play with your ellipse for a while. **Check** that you can move points A and F_1 and that the rest of your ellipse “moves with them”.

 Move focus F_1 until it's almost on top of the origin. How does that affect your ellipse?

 Move focus F_1 until it's almost on top of point A . How does that affect your ellipse?

 The eccentricity of an ellipse is defined as the ratio c/a , i.e. $\frac{\text{distance from center to focus}}{\text{semimajor axis}}$. Move points F_1 and A until your ellipse has eccentricity $3/5$ (you can have GSP compute this for you) and then compute the ratio $\frac{\text{minor axis}}{\text{major axis}}$.

 Can an ellipse have eccentricity > 1 ? What is an ellipse with eccentricity 0? In general, how does the eccentricity of an ellipse relate to its shape?

We want to investigate reflections inside ellipses. The first step is to construct an appropriate ray.


6. Construct a point P on the upper half of your ellipse and construct ray (not segment) $\overline{F_1P}$. This ray will represent a ray from focus F_1 reflecting off the ellipse.


To construct the reflection, we will have to make a tangent line on the ellipse. Instead of constructing a tangent, we'll construct an almost-tangent, i.e. a secant that passes through two points in close proximity. The next two steps show how to do that.


7. Select P and measure its x -coordinate (should be called x_P). Create a new parameter, call it h , and let $h = 0.5$. Then make a new calculation of $x_P + h$, and label it x_Q (tip: you can make a subscript by typing **x[Q]** in the dialog box). Compute $f(x_Q)$ by going to Measure→Calculate, selecting $f(x)$, and then selecting x_Q . Then select x_Q and $f(x_Q)$ in that order and go to Graph→Plot as (x,y) . Check that your new point is actually on the ellipse, and that it moves when P moves.
8. Construct the line (not just the segment) between P and Q . Then double-click on h and change its value to 0.01. [Why didn't we make it 0.01 in the first place?]

Almost done. Now we have to reflect the “ray” in the tangent line.

9. Select the almost-tangent at P and go to Transform→Mark Mirror. Then select the vertical line through P (this is the incoming ray) and go to Transform→Reflect.


 Animate point P on the ellipse. What about the reflected ray is invariant?


 Adjust your foci until your ellipse is almost circular. What property of reflections in circles is related to the invariant you found above?

 What happens to your ellipse if you move the focus *past* the major axis endpoint?

When Mr. Karafiol built this sketch at home, he used a slightly different formula for his ellipse, namely, $f(x) =$

$\sqrt{(a^2 - c^2)\left(1 - \frac{x^2}{a^2}\right)}$, and $g(x) = -f(x)$. Change your formula to match his.

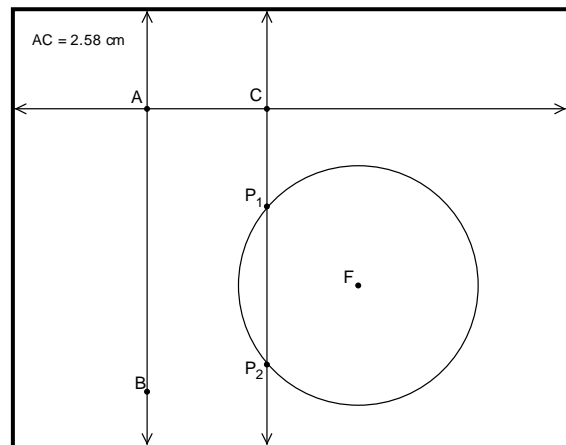
 Does the ellipse's shape change when you change the formula? Why or why not?

 Now move the focus out past the major axis endpoint. What happens to the ellipse now?

Part II—Parabolas and Beyond

The classical definition of a parabola is “the set of points equidistant from a given point (the focus) and a line (the directrix).” We can use that definition to construct a parabola (look ma, no coordinates!).

1. Make a new sketch. Draw a line (roughly vertical) and plot a point not on the line. Label the points on the line A and B , and label the other point F .
2. Select point A and line \overline{AB} and construct a perpendicular line. Construct a new point on the perpendicular line, call it C , and construct a line through C parallel to \overline{AB} .
3. Select points A and C and go to **Measure**→**Distance**. Then select point F and the distance measurement and go to **Construct**→**Circle by center and radius**.
4. Adjust point C until the circle and the line through C intersect twice. Label those points of intersection P_1 and P_2 . Then **trace** them and animate point C .



The diagram should look like the one at right.

As C moves closer to line \overline{AB} , what happens to points P_1 and P_2 ?

Explain why the distance from P_1 to line \overline{AB} is guaranteed to be equal to the distance from P_1 to point F .

5. While tracing and animating is kind of cool, it would be nice if we could just see the whole curve at once. Erase the traces (and stop tracing); select points C and P_1 and go to **Construct**→**Locus**. Repeat with C and P_2

→ “Locus” is just a fancy way of saying “the set of points with a certain property.” So we could have said earlier “A parabola is the locus of points equidistant from a given point and a given line.”

How does the parabola change if you move F further away from \overline{AB} ? Closer?

Now we’re going to extend the definition of a parabola a little bit. We’ll say that a shape is a *pseudoparabola* if it is the locus of points such that the ratio $\frac{\text{distance to Focus}}{\text{distance to Directrix}} = \text{constant}$. (For a parabola, that constant equals 1.)

6. Go to **File**→**Document Options**. Rename your page “Parabola”, then go to **Add Page**→**Duplicate**→**Parabola**. Name the new page “Pseudoparabola”.
7. Delete the circle around F . Go to **Graph**→**Create New Parameter**, and call it e . Right-click on e and make its domain 0 to 4, increasing at 0.25 per second. Its initial value should be 1.
8. Go to **calculate** and calculate $e \cdot AC$. Construct a circle centered at F with this value as its radius.
9. Repeat steps 3-5 above to first trace, then get the locus.

Adjust e to 0.5. What shape is created?

Now animate parameter e . What shapes do you see? How does the shape relate to the value of e ?