ARML D-Team
Problem Set 0

Instructions: You should give each of these problems a good try before consulting an expert (your math teacher, your friend on A or B team, whatever), but you should feel free to discuss these problems with non-experts (friends on D team, other math team members, etc.) while you work on them. (So: talk while you're working if you're talking to someone of your own level of expertise; wait until you're really stuck before talking to an expert.)
NO CALCULATORS ARE ALLOWED.

Please write up careful *solutions* that show not just your answers but also your justifications for them. *If you cannot solve a problem, don't stress, but do include the work you have done and any ideas you have, so that we can see how you are thinking about it.* Solutions (again, not just answers) are due at our first practice, March 26. Read the homework policy if you have other questions about what we want.

1. Consider the circle $x^2 + y^2 = 1$ and the line $y = mx - 2$. Find the value of m that makes the line tangent to the circle.

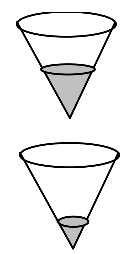
2. Let $d(n)$ be the number of diagonals in a regular n-gon. Find n such that $d(d(n)) = 6902$.

3. A drawer contains both red and blue socks. When two socks are drawn at random, the probability that both are red is exactly 1/2. What is the smallest possible number of socks in the drawer?

4. A triangle with sides $a \le b \le c$ is <u>log-right</u> if $\log(a^2) + \log(b^2) = \log(c^2)$. Compute the largest possible value of a in a right triangle that is also log-right.

5. Describe four different ways of dissecting a given generic (not isosceles, right, or any other particularly special) triangle into four regions of equal area. A "dissection" cuts the triangle into separate non-overlapping regions that can be assembled into the original triangle.

6. Two identical right circular cones, each of height 2, are placed as shown. At the start, the upper cone is full of water and the lower cone is empty. Then water drips down into the lower cone through a hole in the apex of the upper cone. Compute the height of water in the lower cone at the moment when the height of water in the upper cone is 1.



7. If the product $(2^{51} + 1)(2^{50} - 1)$ is computed in base 2, compute the number of 0's in the result.

8. If a, n, and k are integers such that $n = ak$, and $1 < k < n$, then a is a *proper divisor* of n. Find all numbers less than 100 which are equal to the *product* of (all) their proper divisors.

9. $ABCD$ is a parallelogram. E is on $\overline{AB}$, and C is on $\overline{FG}$, so that $DEFG$ is also a parallelogram. $\frac{AE}{EB} = \frac{1}{5}$ and $\frac{FC}{CG} = \frac{1}{3}$. If the area of $ABCD$ is 17 square units, find the area of $DEFG$.

10. Let $F_1, F_2, \ldots$ be the Fibonacci sequence 1,1,2,3,5,…. Prove that, for all positive integers n,

$$F_1 + F_3 + \ldots + F_{2n-1} = F_{2n}.$$

11. Equilateral triangle ABC is inscribed in circle O. Let D and E be midpoints of $\overline{AC}$ and $\overline{AB}$, and let $\overrightarrow{DE}$ intersect the circle at F. If $\frac{DE}{EF}$ can be written as $\frac{a+\sqrt{b}}{c}$, for integers a, b, and c, in simplest form, compute the ordered triple (a, b, c).

12. Compute $\tan x$ if $\frac{\sin^2 x}{3} + \frac{\cos^2 x}{7} = \frac{-\sin(2x)+1}{10}$.